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Exact Results at the 2-D Percolation Point

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Abstract

We derive exact expressions for the excess number of clusters b and the excess cumulants b_n of a related quantity at the 2-D percolation point. High-accuracy computer simulations are in accord with our predictions. b is a finite-size correction to the Temperley-Lieb or Baxter-Temperley-Ashley formula for the number of clusters per site n_c in the infinite system limit; the b_n correct bulk cumulants. b and b_n are universal, and thus depend only on the system's shape. Higher-order corrections show no apparent dependence on fractional powers of the system size.

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Percolation is perhaps the simplest non-trivial model in statistical mechanics. A broad array of techniques have been brought to bear on it by researchers in a variety of disciplines, and it has continued as an active research area for many years (for reviews, see [1,2]). In this work, we restrict ourselves to the case of two-dimensional systems at the percolation point p_c . Even this limited subject has been and continues to be studied via renormalization group ([3] and references therein), conformal field theory [4–6], Coulomb gas methods [7,8], computer simulation ([9,10] and references therein), as an example of supersymmetry [11], and using rigorous mathematical methods (see [1] and references below).

In this Letter we consider the average number of clusters (connected graphs) $\langle N_C \rangle$ on a lattice of N sites and in addition the cumulants C_n of a related quantity, $N_C + N_B/2$, where N_B is the number of bonds. We derive new exact results for the excess or finite-size correction term b to the former and b_n to the latter. By use of conformal field theory and Coulomb gas methods explicit expressions for both quantities on a torus (including the cylinder limit), or a rectangle (with conformally invariant boundary conditions) are determined. These expressions depend only on the aspect ratio of the geometry in question. They are universal, and therefore apply to any particular realization, e.g. lattice and percolation type. For bond percolation on a square (triangular) lattice, the formula for n_c , the number of clusters per site in the infinite system limit, is known from the work of Temperley and Lieb [12] (Baxter, Temperley and Ashley [13]). Thus b represents the finite-size correction to either of these results. However, being universal, its validity is more general and applies to any system at criticality. For the torus or cylinder, where b is the leading finite-size correction, our formulas agree extremely well with the results of computer simulations.

Our work was motivated by the results of [10], who found numerically that b for critical percolation in large systems appears to be a universal quantity, dependent upon the aspect ratio r but independent of the type of lattice or percolation being considered. Their values of $b(r)$ are plotted in Fig. 1 below, while Table I presents new results for the infinite aspect-ratio limit, where the torus becomes a cylinder and $b(r) \rightarrow \tilde{b}r$. Here we find the identical value $\tilde{b} = 0.3608(1)$ for bond percolation on both square and triangular lattices, which

demonstrates the universality of this quantity, and agrees with the conformal prediction to all figures.

Hu and Lin have studied the universality of a related but different quantity, the number of percolating clusters, via Monte Carlo simulation [14]. Some recent theoretical results for this quantity are given by Cardy [15].

We also report (Table I) new results for a few excess cumulants b_n of the quantity $N_C + N_B/2$, specifically b_2 , which corrects the fluctuation, b_3 , which enters in the excess skewness and b_4 , which is related to the excess kurtosis. For the square lattice, the bulk value of the fluctuation is given by Temperley and Lieb [12]. For percolation on a torus, we find non-monotonic behavior for b_2 as a function of aspect ratio. Good agreement with computer results for b_2 , b_3 , and b_4 is obtained.

New computer simulations were carried out to investigate the behavior of $\langle N_C \rangle$ and C_n , $n = 2, 3, 4$, in the cylindrical limit (high aspect ratio torus). We used tori of dimensions $L \times L'$ with $L'/L \geq 32$ and $L = 4, 5, 6, 7, 8, 10, 12, 16$, with additional runs at $L = 32, 64, 128, 256$ where we measured $\langle N_C \rangle$ only. About 10^{12} lattice points were generated for each lattice size, requiring a total of several months of workstation computer time. Such extensive work was necessary to pinpoint the higher-order corrections to b and the higher cumulants. With these aspect ratio ≥ 32 , it was verified that no cluster percolated the long way around the torus, so infinite cylinders were effectively represented. Most of our calculations were done for bond percolation on a square lattice. We also measured $\langle N_C \rangle$ for bond percolation on the triangular lattice, which can also be wrapped around to make a simple cylinder, in order to check the universality hypothesis. In addition, new simulations for $L \times L$ systems were carried out.

Our results solidify the identification of critical percolation as a realization of conformal field theory. This connection was first noticed some years ago [4] and made particularly explicit by Cardy [5], who studied the crossing probabilities via correlation functions of boundary operators. Here we operate at a simpler level, employing the partition function. Thus our results may suggest a way to a deeper understanding of this identification — an

independent derivation of b , for instance, would be very interesting. There is in fact recent progress of this type for the crossing probability in Voronoi percolation [16].

At the critical point, the partition function of the Q -state Potts model, for Q real, may be expressed in random cluster form (see [17])

$$Z = \sum_{\text{graphs}} Q^{N_C + N_B/2} \quad (1)$$

where the sum extends over all graphs (possible placements of N_B bonds on the edges of a lattice), N_C is the number of clusters (connected vertices, including isolated points), and the coupling has been set to unity. The transition is second-order for $0 \leq Q \leq 4$, with critical temperature given by $e^{\beta_c} - 1 = Q^{1/2}$. Here we make use of the $Q = 1$ case, where the graphs are equally weighted, so that Z describes percolation [18]. Similar results may be obtained for the Ising ($Q = 2$) and other universality classes; these will be reported elsewhere.

The cumulants C_n of $N_C + N_B/2$ follow immediately,

$$C_n = \left(Q \frac{d}{dQ} \right)^n \ln Z \quad (2)$$

Note that $C_1 = \langle N_C + N_B/2 \rangle$, C_2 gives the corresponding fluctuation, C_3 enters in the skewness, etc.

At the critical point, Z is supposed to factorize

$$Z = \hat{Z} \cdot Z_u \quad (3)$$

where \hat{Z} is non-universal, depending on the lattice type, boundary conditions, etc., while Z_u encodes the universal information. The corresponding universal term $F_u = -\ln Z_u$ is the finite-size correction to the free energy; it also generates the excess quantities studied here. For a given geometry, it depends only on the central charge

$$c = 1 - \frac{6 \left[\cos^{-1}(\sqrt{Q}/2) \right]^2}{\pi \left[\pi - \cos^{-1}(\sqrt{Q}/2) \right]} \quad (4)$$

[7] and the shape.

The simplest case is an infinite cylinder of width (circumference) L . Here, if we consider a length L' of the cylinder, F_u is given by [19,20]

$$F_u = -\frac{\pi c}{6} \frac{L'}{L} \quad (5)$$

For an infinite strip with edges (and conformally invariant boundary conditions), the formula is the same, except that the 6 is replaced by 24 [19].

Now for percolation ($Q = 1$, $c = 0$) on a cylinder (or torus) the graphs are equally weighted so the number of bonds is exactly proportional to the number of lattice sites. Thus there is no finite-size correction to $\langle N_B \rangle$. The above then leads to $C_1 - \langle N_B \rangle/2 = \langle N_C \rangle = n_c LL' + \tilde{b} L'/L$ ($L' \gg L$), with

$$\tilde{b} = \frac{5\sqrt{3}}{24} = 0.360\,844\dots \quad (6)$$

The agreement with computer simulation results is excellent, as shown in Table I. For the strip (with edges), the r.h.s. of (6) is divided by four.

For a torus, the universal factor Z_u (for arbitrary Q) may be evaluated via the work of di Francesco, Saleur and Zuber [7], who employ a Coulomb gas formulation. This results in an expansion in powers of the parameter $q = e^{-2\pi r}$, where $r = L'/L$ (length/width) of the torus. For percolation, the above procedure then leads to $\langle N_C \rangle = n_c LL' + b$, with

$$\begin{aligned} b = & \frac{5\sqrt{3}}{24} r + q^{5/4} \left(-\frac{1}{2} + 2\sqrt{3}r \right) + q^2 \left(-1 + \sqrt{3}r \right) \\ & + \dots + q^{5/48} + 2q^{53/48} - q^{23/16} + q^{77/48} + \dots, \end{aligned} \quad (7)$$

keeping all terms through $O(q^2)$. The terms proportional to r appear because the powers of q in Z_u depend on Q , and thus contribute to the derivative. They include the leading result (6) for $r \rightarrow \infty$, which arises from the derivative of the partition function of the unit operator. Note that the term correcting Z_u for “cross-topology” clusters is proportional to $Q - 1$ (see [7] (4.8)), and thus does not contribute this way, since $Q = 1$ here. Fig. 1 compares (7) with computer simulations. Excellent agreement is found again.

By symmetry, b is the same for r and $1/r$, although this is not evident from the few terms exhibited in (7). A graph of b vs. r shows that the leading behavior in the cylinder limit (the first term on the r.h.s.) is corrected by a positive term that goes smoothly to zero as r increases from 1. This term is such that $r = 1$ is the only minimum of b .

For a rectangle, with edges, of length L and width L' , Z_u is known (for arbitrary c) via the results of Kleban and Vassileva [21]. One can therefore evaluate F_u for any Q by the same procedure. For percolation one has (here, the finite-size correction to $\langle N_B \rangle$ is independent of L'/L , and is therefore not included)

$$b = \frac{5\sqrt{3}}{32\pi} \ln LL' - \frac{5\sqrt{3}}{16\pi} \ln[\eta(q)\eta(\tilde{q})], \quad (8)$$

where η is the Dedekind η -function, $q = e^{-2\pi r}$ with $r = L'/L$, and $\tilde{q} = e^{-2\pi/r}$. However, we have not attempted computer simulations in this case (or for the strip with edges), since the leading finite-size correction will be due to a non-universal edge term, making the precise determination of b or b_n difficult. Note that the Q -dependence of b_n , for any n , involves the same function of q shown in (8), by contrast to the torus. This is related to the fact that Z on a rectangle generally couples only to the unit operator [21]. Eq. (8) also assumes there is no Q -dependence of the matrix element or constant term in F_u (see [21] for details).

The fluctuation in $N_C + N_B/2$ is given by C_2 . For percolation on a cylinder, its finite-size correction $\tilde{b}_2 L'/L$ follows from (5). We obtain

$$\tilde{b}_2 = \frac{5\sqrt{3}}{36} - \frac{9}{16\pi} = 0.061\,513\dots \quad (9)$$

Computer simulations results for this quantity are illustrated along with (5) in Table I.

The excess fluctuation of the number of clusters N_C would also be of interest. However, to calculate this requires the excess for $\langle N_C N_B \rangle$, which is apparently not accessible. A similar situation occurs for the leading fluctuations in the Temperley-Lieb theory [10,12].

For \tilde{b}_3 and \tilde{b}_4 we find, similarly from (2)–(5), $\tilde{b}_3 = 5\sqrt{3}/36 - 9/8\pi + 27\sqrt{3}/64\pi^2 = -0.043\,500\dots$ and $\tilde{b}_4 = 25\sqrt{3}/108 - 9/4\pi + 27\sqrt{3}/16\pi^2 - 81/64\pi^3 = -.059\,933\dots$. Good agreement with simulations is obtained here as well; see Table I. These values are obtained by linear fits to the data vs. L^{-2} , which also yields the bulk values of C_2 , C_3 and C_4 . Our results for C_2 and C_3 are in agreement with the predictions $0.039\,445\dots$ and $0.012\,913\dots$ that follow from [12]; we have not been able to obtain C_4 either analytically or numerically from that theory, but find from our simulations $C_4 \approx 0.0028$. Details will be presented

elsewhere. The \tilde{b}_n results are less accurate than those for \tilde{b} because of the problem of extrapolating to infinity, and because the error grows rapidly as n increases (since higher moments are involved).

The above also gives the excess fluctuations b_2 for percolation on a torus. One finds

$$\begin{aligned} b_2 = & \tilde{b}_2 r + q^{5/48} \left(1 - \frac{11\sqrt{3}}{48} r \right) - q^{5/24} \\ & + q^{53/48} \left(2 - \frac{11\sqrt{3}}{24} r \right) - 4q^{29/24} \\ & + q^{5/4} \left(\frac{1}{2} + \left[\frac{53\sqrt{3}}{24} - \frac{9}{4\pi} \right] r + \frac{3}{2} r^2 \right) + \dots \end{aligned} \quad (10)$$

keeping terms through $O(q^{5/4})$ to show the appearance of r^2 in the higher coefficients. At $r = 1$, $b_2 = 0.105\,436\,634\dots$ is at a local maximum (note $b_2(r) = b_2(1/r)$). As the aspect ratio r increases, b_2 decreases to a minimum of $0.103\,341\,600\dots$ at $r \approx 1.6177$. As r increases further, b_2 increases, with limiting (large r) behavior agreeing with (9). Such non-monotonic behavior with aspect ratio is very reminiscent of the finite-size correction to the specific heat (which is itself proportional to the excess of the fluctuation of N_B) in the Ising model critical region. This also exhibits non-monotonic behavior as a function of r [22] (see their Fig. 6). This curious feature has been explained in terms of a one-dimensional array of domain walls [23]; a connection to whatever mechanism is operating at the percolation point would be very interesting.

We have also considered the convergence of the excess quantities. This is equivalent to studying the higher order finite-size corrections to the density or cumulants. In all cases examined, the leading such term is proportional to L^{-2} ; however its coefficient is definitely not universal. Fig. 2 illustrates this for $b(r = 1)$. For the square lattice on the cylinder, assuming the exact n_c and \tilde{b} , an analysis of the higher-order corrections shows no sign of a fractional power of L , signifying an irrelevant singularity in the renormalization group, as are seen in some other problems in percolation [3], and suggests a simple analytic series

$$\frac{\langle N_C \rangle}{L^2} = n_c + \frac{\tilde{b}}{L^2} + \frac{0.180}{L^4} + \frac{0.69}{L^6} + \dots \quad (11)$$

Eq. (11) fits all our measurements of $\langle Nc \rangle$ to nearly their full accuracy, about $\pm 5 \cdot 10^{-7}$. This form is consistent with corrections to scaling due to the breakdown of rotational invariance on a square lattice [24].

In summary, we have shown that at the percolation point in two dimensions, the universality of the excess cluster number b , observed to hold numerically in [10], follows from conformal field theory. We have derived explicit expressions for b and also for the excess cumulants of a related quantity that are in complete agreement with simulation results.

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FIGURES

FIG. 1. Excess number of clusters b vs. aspect ratio r at the percolation point. Solid line: exact results; points: simulations [10]. The simulations combine several types of percolation and lattice types. The error is less than the size of the points.

FIG. 2. $b_{\text{eff}}(r = 1)$ vs. L^{-2} . Points: simulation results for an $L \times L$ square lattice, Solid line: fitting curve $0.88358 + 0.18L^{-2}$ consistent with (11). b_{eff} is defined as $\langle N_C \rangle - n_c L^2$, where $n_c = (3 \cdot 3^{1/2} - 5)/2$ [10,12]. The conformal prediction for $b(r = 1)$ from (7) is $0.883576\dots$

TABLES

TABLE I. Exact and simulated results for excess cumulants at the percolation point. See text for analytic expressions.

Quantity	Exact	Simulation
\tilde{b}	0.360 844...	0.3608(1)
$b_2(r = 1)$	0.105 437...	0.106
\tilde{b}_2	0.061 513...	0.068
\tilde{b}_3	-0.043 500...	-0.041
\tilde{b}_4	-0.059 933...	-0.059



